

Discrete Cycloids from Convex Symmetric Polygons

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Abstract Cycloids, hypocycloids and epicycloids have an often forgotten common property: they are homothetic to their evolutes. But what if use convex symmetric polygons as unit balls, can we define evolutes and cycloids which are genuinely discrete? Indeed, we can! We define *discrete cycloids* as eigenvectors of a discrete *double evolute transform* which can be seen as a linear operator on a vector space we call *curvature radius space*. We are also able to classify such cycloids according to the eigenvalues of that transform, and show that the number of *cusps* of each cycloid is well determined by the ordering of those eigenvalues. As an elegant application, we easily establish a version of the four-vertex theorem for closed convex polygons. The whole theory is developed using only linear algebra, and concrete examples are given.

Keywords Cycloids · Discrete Evolute · Four-Vertex Theorem · Minkowski Geometry

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1 Introduction

Euclidean cycloids (and hypocycloids and epicycloids) can be characterized as planar curves which are λ -homothetic to their evolutes (which kind of cycloid depends on the signal of $\lambda - 1$). Such idea can naturally be extended to normed planes (see [2]) if the unit ball is sufficiently smooth. But what if the ball is a polygon? Can we define genuinely discrete analogues of cycloids without using limiting processes?

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In this paper, we present a discrete evolute transform and then define discrete cycloids as polygonal lines which are homothetic to their double evolutes. Our evolute construction is an alternative to [1] – we follow instead the approach in [3]. By using a suitable representation, we can represent our polygonal lines as vectors in a space we call L_P . In this space, the double evolute transform is linear, so we rephrase the problem in two different ways: as an eigenvector problem, or as a recurrence. By analysing the interaction of this transform within many subspaces of L_P , we are able to establish the spectral representation of the double evolute transform. As a consequence, we are able to decompose any element of L_P as a sum of cycloids, producing a generalization of the Discrete Fourier Transform. As an application of this decomposition, we provide an elegant proof of a four vertex theorem for polygons.

We provide explicit cycloid formulae when the polygon is regular; as expected, if the polygon is close to the usual euclidean unit ball, the corresponding cycloids are approximations of the classical cycloids.

More specifically, we start with a symmetric polygon $P = P_1 P_2 \dots P_{2n}$ which will be our unit ball on the plane, and define its dual ball $Q = Q_1 Q_2 \dots Q_{2n}$. A polygonal line M whose sides are respectively parallel to the sides of P can then be represented by the lengths r_1, r_2, \dots of its sides (taking the sides of P as unit length in each direction) – we call this the *curvature radius representation* of M . We show that M is a cycloid exactly when its radii satisfy a difference equation of the kind

$$\frac{-1}{[P_i, P_{i+1}]} \nabla_i \left(\frac{\Delta_i r}{[Q_i, Q_{i+1}]} \right) = \lambda r_i \quad (1)$$

where Δ_i and ∇_i are forward and backward difference operators, and $[\cdot, \cdot]$ is the determinant of two vectors. This equation is the natural discretization of the 2nd order Sturm-Liouville type Differential Equation displayed in [2], so we can reasonably expect it to have a nice spectral structure (as seen in [8]). Indeed, if we require the list r_i to be $2n$ -periodic, we are able to show that the eigenvalues associated to Eq. (1) can be ordered as

$$\lambda_0 (= 0) < \lambda_1^1 = \lambda_1^2 (= 1) < \lambda_2^1 \leq \lambda_2^2 < \lambda_3^1 \leq \lambda_3^2 < \dots \lambda_{n-1}^1 \leq \lambda_{n-1}^2 < \lambda_n$$

where each eigenvector is a closed cycloid – except for those with eigenvalue 1, which form a 2-dimensional space of non-closed cycloids (there are no non-trivial hypocycloids). Moreover, we show that the cycloid associated to λ_k^i has exactly $2k$ ordinary cusps. When k is even, the cycloid is a symmetric polygon; when k is odd ($\neq 1$), the cycloid is a polygon of 0-width. It is interesting to note that, in the Euclidean case, the eigenvalue λ is **determined** by the number of cusps of the cycloids – not the case here, since we might have $\lambda_k^1 \neq \lambda_k^2$!

Figure 1 shows a concrete example, where the unit ball is the octagon in the top image (and the cycloid for $\lambda_0 = 0$). The next row pictures two different open cycloids ($\lambda = 1$). The third row shows symmetric cycloids ($\lambda_2^1 \approx 3.21$ and $\lambda_2^2 \approx 3.46$), each with 4 cusps. The fourth row shows 0-width cycloids ($\lambda_3^1 \approx 4.90$ and $\lambda_3^2 \approx 7.92$), with 3 "double" cusps each. Finally, we have one last cycloid ($\lambda_4 \approx 8.15$) where all 8 vertices are cusps.

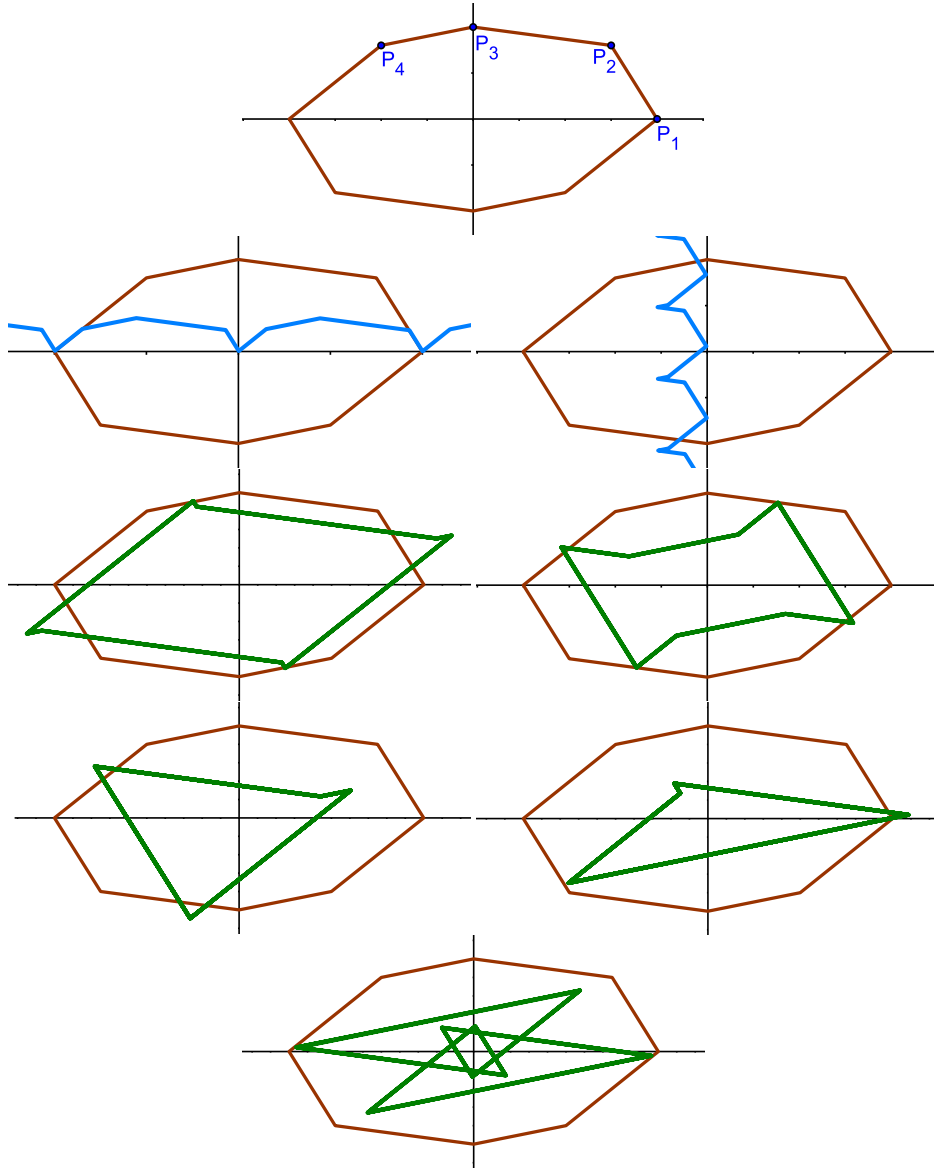


Fig. 1 Cycloids related to an octagon

If we require instead the list r_i to be just $2mn$ -periodic, the old eigenvalues are joined by new ones in a very orderly fashion

$$\begin{array}{ccccccc}
 & & & & & \lambda_0 (= 0) & < \\
 < \lambda_{1/m}^1 = \lambda_{1/m}^2 & < \lambda_{2/m}^1 = \lambda_{2/m}^2 & < \dots < \lambda_{1-1/m}^1 = \lambda_{1-1/m}^2 < \lambda_1^1 = \lambda_1^2 (= 1) < \\
 < \lambda_{1+1/m}^1 = \lambda_{1+1/m}^2 & < \lambda_{1+2/m}^1 = \lambda_{1+2/m}^2 & < \dots < \lambda_{2-1/m}^1 = \lambda_{2-1/m}^2 < \lambda_2^1 \leq \lambda_2^2 < \\
 < \lambda_{2+1/m}^1 = \lambda_{2+1/m}^2 & < \lambda_{2+2/m}^1 = \lambda_{2+2/m}^2 & < \dots < \lambda_{3-1/m}^1 = \lambda_{3-1/m}^2 < \lambda_3^1 \leq \lambda_3^2 < \\
 < \dots & \dots & \dots & \dots & \dots & \dots & < \\
 < \lambda_{n-1+1/m}^1 = \lambda_{n-1+1/m}^2 < \lambda_{n-1+2/m}^1 = \lambda_{n-1+2/m}^2 < \dots < \lambda_{n-1/m}^1 = \lambda_{n-1/m}^2 < \lambda_n^1 < \lambda_n^2
 \end{array}$$

(note the previous eigenvalue list in the last column – they **do not** necessarily come in identical pairs). In summary, we have $2m-2$ non-trivial hypocycloids, 2 cycloids and $2mn-2m-1$ epicycloids. The fractional indices work across different periods – for example, the eigenvalue $\lambda_{2/5}^1$ which appears when looking for a list of period $10n$ is indeed the same as $\lambda_{6/15}^1$ which would show up when looking for period $30n$.

Many of the results we present have been previously established in a more general context – see [8], for example – but our approach is very geometric and requires only basic Linear Algebra.

The organization of the paper is as follows: in section 2 we establish basic facts about *the polygonal unit ball and its dual*. Section 3 presents the *curvature radius space* (with an inner product) and establishes geometric interpretations of many of its interesting subspaces. In section 4 we present the discrete *evolute* and the *double evolute* transforms. Section 5 finally defines the *cycloids*, starting out with the important example when the ball is a regular polygon, and proceeds to the spectral analysis of the double evolute transform and the determination of the number of cusps of each cycloid (in the case where the cycloids have the same period as the unit ball). In section 6, we show that the general case where our cycloids have *other periods* than the unit ball can be reduced to the $2n$ -periodic polygons, establishing the more general result. Finally, in section 7 we use our framework to quickly prove a four vertex theorem for polygons.

1.1 Notation

Given two points or vectors $v, w \in \mathbb{R}^2$, we denote the *determinant* whose columns are v and w by $[v, w]$. Given a list of numbers (or vectors) $\{L_i\}_{i \in \mathbb{Z}}$, we define the *forward difference operator* $\Delta L = \{\Delta_i L\}_{i \in \mathbb{Z}}$ where $\Delta_i L = L_{i+1} - L_i$ and the *backward difference operator* $\nabla L = \{\nabla_i L\}_{i \in \mathbb{Z}}$ where $\nabla_i = \Delta_{i-1}$ (whenever necessary, indices are taken mod $2n$); their composition will be denoted δ^2 (so $\delta^2 L = L_{i+1} - 2L_i + L_{i-1}$). Whenever two matrices A and B are similar (that is, $A = M^{-1}BM$ for some invertible matrix M), we write $A \approx B$.

2 The polygonal ball and its dual

We start out with a plane symmetric non-degenerate convex polygon $P = P_1 P_2 \dots P_{2n}$ (that is, $P_{i+n} = -P_i$), which will be our reference *ball of radius 1* – we might as well imagine that the vertices are in counter-clockwise order around the origin¹. Its *dual* is the only polygon $Q = Q_1 Q_2 \dots Q_{2n}$ which satisfies

$$[P_i, Q_i] = 1 \text{ and } [\Delta_i P, Q_i] = [P_{i+1}, \Delta_i Q] = 0$$

for $i = 1, \dots, 2n$. Note that, in view of the first equation, the two last ones are actually equivalent, since

$$\Delta_i [P_i, Q_i] = [\Delta_i P, Q_i] + [P_{i+1}, \Delta_i Q] = 0.$$

Geometrically, each vector Q_i is parallel to the corresponding side of P , and vice-versa. These conditions actually allow us to write explicitly

$$P_{i+1} = -\alpha_i \Delta_i Q \text{ and } Q_i = \beta_i \Delta_i P \tag{2}$$

¹ Choosing a convex symmetric unit ball is the same as choosing a norm in the plane, so we are in the realms of Minkowski Geometry [7].

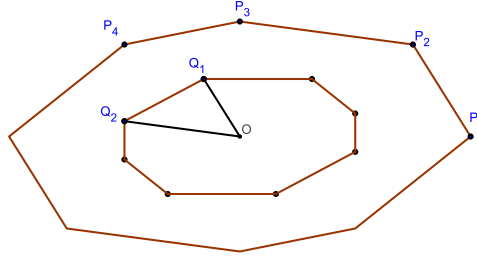


Fig. 2 A P -ball and its dual

where

$$\alpha_i = \frac{1}{[Q_i, Q_{i+1}]} \text{ and } \beta_i = \frac{1}{[P_i, P_{i+1}]} \quad (3)$$

From there, it is easy to see that the dual ball is also symmetric. Moreover, since

$$\begin{aligned} [Q_{i-1}, Q_i] &= \beta_{i-1} \beta_i [\nabla_i P, \Delta_i P] > 0 \\ [\nabla_i Q, \Delta_i Q] &= [[Q_{i-1}, Q_i] P_i, [Q_i, Q_{i+1}] P_{i+1}] = \beta_{i-1} \beta_i \beta_{i+1} [\nabla_i P, \Delta_i P] [\Delta_i P, \Delta_{i+1} P] > 0 \end{aligned} \quad (4)$$

we see that Q is also convex and ordered in a counter-clockwise orientation (in particular, all α_i and β_i are positive). Finally, we note for future reference that any scaling of a factor γ applied to P implies in a scaling of factor $\frac{1}{\gamma}$ applied to Q .

3 The curvature radius space

Consider all polygonal lines $M = (M_i)_{i \in \mathbb{Z}}$ whose sides are respectively parallel to the corresponding sides of P (though they do not necessarily close, let us call them P -polygons anyway). More explicitly, we will require

$$\Delta_i M = r_i \cdot \Delta_i P$$

for some list of real numbers $(r_i)_{i \in \mathbb{Z}}$ (we allow $r_i = 0$ with no further ado). Each number r_i will be called the *curvature radius* of the side $M_i M_{i+1}$ (with respect to P). Up to a translation, all P -polygons are uniquely represented by the list (r_i) .

Before continuing, we add another restriction – we require the sides (but not necessarily the vertices!) to repeat:

Definition 1 A polygonal line M is a *periodic P -polygon* when

$$\Delta_{i+2n} M = \Delta_i M$$

for all $i \in \mathbb{Z}$. Since in this case we clearly have

$$r_{i+2n} = r_i \quad (5)$$

each such polygonal line M (up to a translation) can be represented by its *radii vector* $\mathbf{r} = (r_i)_{i=1,2,\dots,2n}$. We write L_P for such space of all periodic P -polygons.

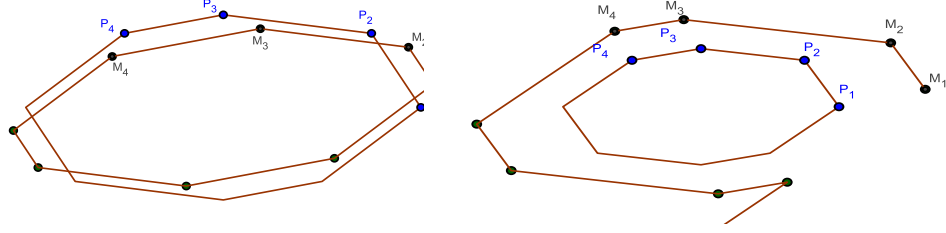


Fig. 3 While $\mathbf{r} = (\frac{1}{2}, 1, \frac{3}{2}, 1, \frac{1}{2}, 1, \frac{3}{2}, 1)$ closes the line, $\mathbf{r} = (1, 2, 1, 2, 1, 2, 1, -1)$ does not

Our goal in this section is to pair up algebraic properties of $\mathbf{r} \in \mathbb{R}^{2n}$ with geometric properties of M (compare this to the similar analysis done in [3]). We start defining a suitable inner product in L_P :

Definition 2 Given two radii vectors \mathbf{r} and \mathbf{s} in L_P , we define their P -inner product as

$$\langle \mathbf{r}, \mathbf{s} \rangle_P = \sum_{i=1}^{2n} \frac{r_i s_i}{\beta_i}$$

Some interesting subspaces of L_P are listed below:

- C_P : the space of all **closed** polygons M . Given our periodicity condition on the sides, it is enough to check if $M_{2n+1} = M_1$, that is

$$\sum_{i=1}^{2n} \Delta_i M = \mathbf{0}$$

or, in terms of the radii

$$\sum_{i=1}^{2n} r_i \Delta_i P = \mathbf{0} \tag{6}$$

Since this last condition is linear, C_P is indeed a subspace of L_P , and $\dim C_P = 2n - 2$ (since there must be two linearly independent $\Delta_i P$).

- S_P : the space of all **symmetric** polygons M . Choosing the origin as the center of symmetry, this condition translates to $M_{i+n} = -M_i$ for all i , or equivalently

$$r_{i+n} = r_i \quad (i = 1, 2, \dots, n)$$

Clearly (both geometrically and algebraically), $S_P \subseteq C_P$, and $\dim S_P = n$.

- A_P : the space of all **anti-symmetric** P -polygons M , which we define algebraically by the condition

$$r_{i+n} = -r_i \quad (i = 1, 2, \dots, n)$$

It is easy to see that $\dim A_P = n$; actually, under our inner product,

$$A_P = (S_P)^\perp.$$

- D_P : the space of **double** polygons, that is, such that $M_{i+n} = M_i$ for all i . This is equivalent to $M_1 = M_{n+1}$ and $\Delta_i M = \Delta_{i+n} M$ for all i , or

$$\begin{cases} r_{i+n} = -r_i & (i = 1, 2, \dots, n) \\ \sum_{i=1}^n r_i \Delta_i P = 0 \end{cases}.$$

Given the first condition, the second is equivalent to Equation 6 so $D_P = A_P \cap C_P$ and $\dim D_P = n - 2$. In fact, since $D_P \perp S_P$ and $\dim D_P + \dim S_P = \dim C_P$, we also have

$$C_P = S_P \oplus D_P$$

More specifically, D_P is the orthogonal complement of S_P in C_P .

- B_P : the space of all **balls** (homothetic to P), which consists of multiples of the vector $\mathbf{1} = (1, 1, \dots, 1)$. Clearly $\dim B_P = 1$ and $B_P \subseteq S_P$.

In order to further geometrically characterize subspaces of L_P , we turn our attention to:

Definition 3 The *support* associated to the side $M_i M_{i+1}$ is the signed distance h_i from the origin to that side, normalized to have value 1 when the polygon is the P -ball itself. More explicitly:

$$h_i = [M_i, Q_i] = [M_{i+1}, Q_i]$$

The *support function* is the list of values $\{h_i\}_{i \in \mathbb{Z}}$.

Proposition 4 The radii vector depends linearly on the support function. Explicitly,

$$r_i = h_i + \beta_i \nabla_i (\alpha_i \Delta_i h) \quad (7)$$

Proof Just use Eq. 2 a few times:

$$\begin{aligned} \alpha_i \Delta_i h &= \alpha_i [M_{i+1}, \Delta_i Q] = [P_{i+1}, M_{i+1}] \Rightarrow \\ &\Rightarrow \beta_i \nabla_i (\alpha_i \Delta_i h) = \beta_i ([P_{i+1}, \Delta_i M] + [\Delta_i P, M_i]) = \\ &= \beta_i [P_{i+1}, r_i \Delta_i P] + [Q_i, M_i] = r_i - h_i \end{aligned} \quad \square$$

While the support function is not invariant by translations, the *width* of a polygon is:

Definition 5 The P -width w_i of M between the sides $M_i M_{i+1}$ and $M_{i+n} M_{i+n+1}$ is the signed distance between such sides, taking P as the unit reference ball. In other words

$$w_i = h_i + h_{i+n} = [M_i - M_{i+n}, Q_i] = [M_{i+1} - M_{i+n+1}, Q_i]$$

Geometrically, P -polygons of constant P -width are characterized by:

Proposition 6 A P -polygon M has constant P -width if and only if each of its “major” diagonals is parallel to the corresponding major diagonal of P , that is,

$$[M_i - M_{i+n}, P_i] = 0 \quad (i = 1, 2, \dots, n).$$

A P -polygon M has constant P -width 0 if and only if

$$M_{i+n} = M_i \quad (i = 1, 2, \dots, n).$$

In other words, D_P is the space of P -polygons of width 0.

Proof For the first statement, we just need to remember once again Equation 2 and write

$$w_i = w_{i-1} \Leftrightarrow [M_i - M_{i+n}, Q_i] = [M_i - M_{i+n}, Q_{i-1}] \Leftrightarrow [M_i - M_{i+n}, \Delta_{i-1}Q] = 0 \Leftrightarrow [M_i - M_{i+n}, P_i] = 0$$

For the second, just note that

$$\begin{aligned} w_i = 0 &\Rightarrow [M_i - M_{i+n}, Q_i] = 0 \\ w_i = w_{i-1} &\Rightarrow [M_i - M_{i+n}, P_i] = 0 \end{aligned}$$

and, since $[P_i, Q_i] = 1$, these two equations are linearly independent, implying $M_i = M_{i+n}$. \square

Adding support functions is the same as performing a *Minkowski Sum* of the corresponding polygons (see [7], for example). Therefore, the statement $C_P = S_P \oplus D_P$ can be translated as "every closed P -polygon is the (Minkowski) sum of a symmetric polygon and a polygon of 0 width".

Though the support function cannot be determined by the radii, it is easy to see that the width \mathbf{w} depends linearly on \mathbf{r} , since

$$w_i = -[M_{i+n} - M_{i+1}, Q_i] = -\left[\sum_{k=i+1}^{i+n-1} \Delta_k M, Q_i\right] = -\sum_{k=i+1}^{i+n-1} r_k [\Delta_k P, Q_i]$$

In particular, we have

$$\mathbf{w}(\mathbf{r} + \lambda \mathbf{1}) = \mathbf{w}(\mathbf{r}) + \lambda \mathbf{w}(\mathbf{1}) = \mathbf{w}(\mathbf{r}) + 2\lambda \cdot \mathbf{1}$$

since the unit ball has constant width 2. So, if \mathbf{r} describes a polygon with constant width $2\lambda_0$, then $\mathbf{r} - \lambda_0 \mathbf{1}$ will be a polygon with constant width 0! This leads us to define:

- W_P : the space of all **constant-width** P -polygons. Then $W_P = D_P \oplus B_P$, and $\dim W_P = n - 1$. In particular, $W_P \subseteq C_P$.

4 Evolutes and double evolutes

4.1 Evolutes

Given the definition of the curvature radii, we can "fit" a ball P of radius r_i to the side $\Delta_i M$. We can then join the centers of such balls to form a new polygonal line:

Definition 7 The P -*evolute* of M is the polygonal line E whose vertices are

$$E_i = M_i - r_i P_i = M_{i+1} - r_i P_{i+1}. \quad (8)$$

This definition is the discrete version of the evolute in [3]. Now, E is a Q -polygon, since

$$\Delta_i E = (M_{i+1} - r_{i+1} P_{i+1}) - (M_i - r_i P_i) = -\Delta_i r \cdot P_{i+1} = \alpha_i \Delta_i r \cdot \Delta_i Q$$

This means we can represent E by its curvature radii *with respect to* Q , namely

$$s_i = \alpha_i \Delta_i r \quad (9)$$

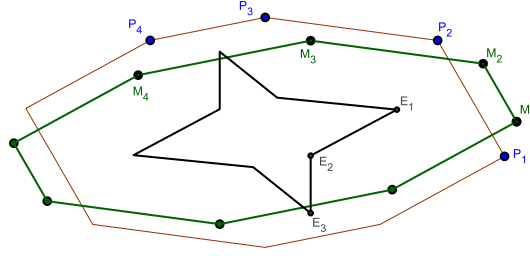


Fig. 4 A polygon M and its P -evolute E

So, using the radius representation, the evolute process $E_P : L_P \rightarrow L_Q$ is a linear transformation, whose $2n \times 2n$ matrix can be explicitly written as

$$E_P = \begin{bmatrix} -\alpha_1 & \alpha_1 & 0 & \dots & 0 & 0 \\ 0 & -\alpha_2 & \alpha_2 & \dots & 0 & 0 \\ 0 & 0 & -\alpha_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\alpha_{2n-1} & \alpha_{2n-1} \\ \alpha_{2n} & 0 & 0 & \dots & 0 & -\alpha_{2n} \end{bmatrix}$$

In order to characterize it geometrically, we need:

Definition 8 Given a periodic P -polygon M represented by the radius vector \mathbf{r} , its *(signed) Q -length* is

$$L_Q(M) = \langle \mathbf{r}, \mathbf{1} \rangle_P = \sum_{i=1}^{2n} \frac{r_i}{\beta_i}$$

which is the signed length of (one period of) M , taking Q as the unit ball, since (from Eq. 2)

$$\Delta_i M = r_i \Delta_i P = \frac{r_i}{\beta_i} \cdot Q_i$$

Similarly, given a periodic Q -polygon N represented by \mathbf{s} , its *(signed) P -length* is

$$L_P(N) = \langle \mathbf{s}, \mathbf{1} \rangle_Q = \sum_{i=1}^{2n} \frac{s_i}{\alpha_i}$$

which is the signed length of N , taking P as the unit ball, since

$$\Delta_i N = s_i \Delta_i Q = -\frac{s_i}{\alpha_i} \cdot P_{i+1}$$

Proposition 9 The image of E_P is the space of all Q -polygons of zero P -length, and its kernel is B_P .

Proof The kernel is easy:

$$s_i = 0 \Leftrightarrow \Delta_i r = 0 \Leftrightarrow r = \lambda \mathbf{1}$$

so we know that $\text{rank}(E_P) = 2n - 1$. Now, the P -length of the evolute E is

$$L_P(E) = \sum_{i=1}^{2n} \frac{s_i}{\alpha_i} = \sum_{i=1}^{2n} \Delta_i r = 0$$

and since the condition $L_P(N) = 0$ determines a subspace of dimension $2n - 1$ as well, we conclude it must be the whole image of E_P . \square

4.2 Double evolutes

But why stop there? If E is a Q -polygon, we can find the evolute of E taking Q as the new reference ball! Namely, we find a new curve F given by:

$$F_{i+1} = E_{i+1} - s_i Q_{i+1} = E_i - s_i Q_i$$

where we shifted the indices in F to compensate for the two forward differences we have taken. We have

$$\Delta_i F = (E_i - s_i Q_i) - (E_i - s_{i-1} Q_i) = -\nabla_i s \cdot Q_i = -\beta_i \nabla_i s \cdot \Delta_i P$$

so F is a P -polygon again! The matrix of this second evolute transform $E_Q : L_Q \rightarrow L_P$ is

$$E_Q = \begin{bmatrix} -\beta_1 & 0 & 0 & \dots & 0 & \beta_1 \\ \beta_2 & -\beta_2 & 0 & \dots & 0 & 0 \\ 0 & \beta_3 & -\beta_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\beta_{2n-1} & 0 \\ 0 & 0 & 0 & \dots & \beta_{2n} & -\beta_{2n} \end{bmatrix}$$

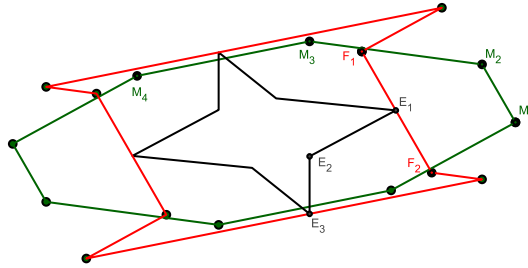


Fig. 5 A polygon M and its double evolute F

Definition 10 With the notation above, F is the *double evolute* of M . Explicitly, F is the curve whose curvature radii are:

$$t_i = -\beta_i \nabla_i s = -\beta_i \nabla_i (\alpha_i \Delta_i r)$$

so the matrix of the *double evolute transform* T_P is

$$T_P = E_Q E_P = \begin{bmatrix} \beta_1 (\alpha_1 + \alpha_{2n}) & -\alpha_1 \beta_1 & 0 & \dots & -\alpha_{2n} \beta_1 \\ -\alpha_1 \beta_2 & \beta_2 (\alpha_2 + \alpha_1) & -\alpha_2 \beta_2 & \dots & 0 \\ 0 & -\alpha_2 \beta_3 & \beta_3 (\alpha_3 + \alpha_2) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\alpha_{2n-1} \beta_{2n-1} \\ -\alpha_{2n} \beta_{2n} & 0 & 0 & \dots & \beta_{2n} (\alpha_{2n} + \alpha_{2n-1}) \end{bmatrix} \quad (10)$$

We quickly note that T_P is invariant by ball rescaling, since a rescaling on P implies in the reverse scaling on Q . We are now ready to justify our choices of inner products:

Proposition 11 For any vectors $\mathbf{r} \in L_P$ and $\mathbf{w} \in L_Q$, we have

$$\langle E_P \mathbf{r}, \mathbf{w} \rangle_Q = \langle \mathbf{r}, E_Q \mathbf{w} \rangle_P$$

that is, $E_Q = E_P^*$ with this choice of inner products.

Proof Just write

$$\begin{aligned} \langle E_P \mathbf{r}, \mathbf{w} \rangle_Q &= \sum_{i=1}^{2n} \frac{\alpha_i \Delta_i r}{\alpha_i} \cdot w_i = \sum_{i=1}^{2n} \Delta_i r \cdot w_i \\ \langle \mathbf{r}, E_Q \mathbf{w} \rangle_P &= \sum_{i=1}^{2n} \frac{r_i}{\beta_i} \cdot (-\beta_i \nabla_i w) = - \sum_{i=1}^{2n} r_i \nabla_i w \end{aligned}$$

and the two sums are just rearrangements of each other. \square

Proposition 12 T_P is self-adjoint (therefore non-negative), and C_P is invariant by T_P .

Proof The first statement follows directly from $T_P = E_P^* E_P$. More explicitly, we have

$$\langle T_P r, r \rangle_P = \sum_{i=1}^{2n} r_i^2 (\beta_i + \beta_{i-1}) - \sum_{i=1}^{2n} 2\beta_i r_i r_{i+1} = \sum_{i=1}^{2n} \beta_i (r_i - r_{i+1})^2 \geq 0 \quad (11)$$

with equality if, and only if, r is a multiple of $\mathbf{1}$ (remember that $\beta_i > 0$ from Eq. 4). The second fact is geometrically clear, but we also offer an algebraic proof: just remember that C_P is defined by $\sum r_i \cdot \Delta_i P = 0$. But the expression on the left side remains invariant under the evolute transform, since

$$\sum r_i \cdot \Delta_i P = - \sum \Delta_i r \cdot P_{i+1} = - \sum \frac{s_i}{\alpha_i} P_{i+1} = \sum s_i \cdot \Delta_i Q. \quad \square$$

5 Discrete cycloids

Definition 13 A *discrete cycloid* is a polygonal line M which is homothetic to its double evolute.

In other words, we want

$$\mathbf{t} = T_P \mathbf{r} = \lambda \mathbf{r},$$

or, in operator notation,

$$\beta_i \nabla_i (\alpha_i \Delta_i r) + \lambda r_i = 0 \quad (12)$$

for some constant λ – an eigenvalue problem!

Example 14 (Regular Polygons) If P is a regular polygon with $2n$ sides, then so is Q . Let $\gamma = \frac{\pi}{2n}$. We might as well rescale P so P and Q are congruent to each other – explicitly, this happens when

$$|P_i| = |Q_i| = \frac{1}{\sqrt{\cos \gamma}} \Rightarrow \alpha_i = \beta_i = \frac{1}{2 \sin \gamma}.$$

So in this case the double evolute transform

$$T_P = \frac{1}{4 \sin^2 \gamma} \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ -1 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}$$

is a discrete convolution! One can verify directly that

$$\mathbf{c}_k = (1, \cos 2k\gamma, \cos 4k\gamma, \dots, \cos 2(2n-1)k\gamma); \quad k = 0, 1, 2, \dots, 2n-1 \quad (13)$$

are eigenvectors, since for $\phi \in \{0, 2k\gamma, 4k\gamma, \dots\}$:

$$2 \cos \phi - \cos(\phi + 2k\gamma) - \cos(\phi - 2k\gamma) = 2(1 - \cos 2k\gamma) \cos \phi = 4 \sin^2 k\gamma \cos \phi$$

In other words, the eigenvalues are

$$\lambda_k = \frac{\sin^2 k\gamma}{\sin^2 \gamma}$$

for $k = 0, 1, 2, \dots, 2n-1$. Since $\lambda_k = \lambda_{2n-k}$, all of them are double eigenvalues, except for $\lambda_0 = 0$ and $\lambda_n = \csc^2 \gamma$. Also, note that $\mathbf{c}_k \in S_P$ if k is even, and $\mathbf{c}_k \in A_P$ if k is odd. In other words, if we rename the pair $(\lambda_k, \lambda_{n-k})$ as $(\lambda_k^1, \lambda_k^2)$ for $k = 1, 2, \dots, n-1$, the eigenvalues can be ordered this way:

$$\lambda_0 (= 0) < \lambda_1^1 = \lambda_1^2 = 1 < \lambda_2^1 = \lambda_2^2 < \lambda_3^1 = \lambda_3^2 < \dots < \lambda_{n-1}^1 = \lambda_{n-1}^2 < \lambda_n (= \csc^2 \gamma)$$

Our goal is now to discover which parts of the spectral structure above remain true in the general case. Before we do that, let us rephrase our problem in a slightly different way – as a recurrence.

5.1 The half-turn transform

Once a candidate value for λ is given, we can see our problem as a recurrence on the coordinates of \mathbf{r} , namely

$$r_{k+1} = \left(1 + \frac{\alpha_{k-1}}{\alpha_k} - \frac{\lambda}{\alpha_k \beta_k}\right) r_k - \frac{\alpha_{k-1}}{\alpha_k} r_{k-1} \quad (k = 2, 3, \dots) \quad (14)$$

or, in matricial form

$$\begin{bmatrix} r_k \\ r_{k+1} \end{bmatrix} = S_k(\lambda) \begin{bmatrix} r_{k-1} \\ r_k \end{bmatrix} \quad \text{where } S_k(\lambda) = \begin{bmatrix} 0 & 1 \\ -\frac{\alpha_{k-1}}{\alpha_k} & 1 + \frac{\alpha_{k-1}}{\alpha_k} - \frac{\lambda}{\alpha_k \beta_k} \end{bmatrix} \quad (15)$$

Since we want a solution $\mathbf{r} \in L_P$, the trick is to find special values of λ , r_1 and r_2 such that $r_{2n+1} = r_1$ and $r_{2n+2} = r_2$ in this recurrence. Actually, in order to separate symmetric from anti-symmetric solutions, we should stop half-way and check the relationship between (r_1, r_2) and (r_{n+1}, r_{n+2}) .

Definition 15 Given $\lambda \in \mathbb{R}$, the *half-turn transform* H_λ is the linear transformation which takes $\boldsymbol{\rho} = (r_1, r_2) \in \mathbb{R}^2$ to $H_\lambda \boldsymbol{\rho} = (r_{n+1}, r_{n+2}) \in \mathbb{R}^2$ according to the recurrence (14) above. In other words

$$H_\lambda = S_{n+1}(\lambda) \cdot S_n(\lambda) \cdot \dots \cdot S_2(\lambda).$$

Though H_λ depends on our index choice (for example, H_λ does **not** necessarily take (r_2, r_3) to (r_{n+2}, r_{n+3})), we note that H_λ **does** take (r_{n+1}, r_{n+2}) to (r_{2n+1}, r_{2n+2}) , because of the n -periodicity of the sequences α and β . Now we can summarize:

- “Finding an eigenvector $\mathbf{r} \in L_P$ of T_P (for the eigenvalue λ)” is equivalent to “finding an eigenvector $\boldsymbol{\rho} \in \mathbb{R}^2$ of H_λ^2 (for the eigenvalue 1)”;
- “Finding an eigenvector $\mathbf{r} \in S_P$ of T_P (for the eigenvalue λ)” is equivalent to “finding an eigenvector $\boldsymbol{\rho} \in \mathbb{R}^2$ of H_λ (for the eigenvalue 1)”;
- “Finding an eigenvector $\mathbf{r} \in A_P$ of T_P (for the eigenvalue λ)” is equivalent to “finding an eigenvector $\boldsymbol{\rho} \in \mathbb{R}^2$ of H_λ (for the eigenvalue -1)”.

Proposition 16 For any $\lambda \in \mathbb{R}$

$$\det H_\lambda = 1$$

This restricts H_λ to six possibilities, according to the **geometric** multiplicity of its eigenvalues:

Case 1. If $H_\lambda = I$, then λ is a double eigenvalue of T_P , and both its eigenvectors are in S_P .

Case 2. If $H_\lambda = -I$, then λ is a double eigenvalue of T_P , and both its eigenvectors are in A_P .

Case 3. If H_λ has a single eigenvalue 1, then λ is a single eigenvalue of T_P , with its eigenvector in S_P .

Case 4. If H_λ has a single eigenvalue -1 , then λ is a single eigenvalue of T_P , with its eigenvector in A_P .

Case 5. If H_λ has two distinct real eigenvalues μ and μ^{-1} , then λ is not an eigenvalue of T_P .

Case 6. If H_λ has two complex eigenvalues $e^{\pm i\theta}$, then λ is not an eigenvalue of T_P .

Proof The determinant follows directly from

$$\det S_k(\lambda) = \frac{\alpha_{k-1}}{\alpha_k}$$

and the symmetry of our ball ($\alpha_{n+1} = \alpha_1$). The existence of the eigenvectors in cases 1-4 follows from the observation before the proposition – we just have to make sure there are no **other** eigenvectors of H_λ^2 in cases 3-6.

In cases 3 and 4, we have

$$H_\lambda \approx \begin{bmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{bmatrix} \Rightarrow H_\lambda^2 \approx \begin{bmatrix} 1 & \pm 2 \\ 0 & 1 \end{bmatrix}$$

which clearly has only one eigenvalue 1. In case 5 the eigenvalues of H_λ^2 are μ^2 and μ^{-2} , and neither is 1. Finally, in case 6:

$$H_\lambda \approx R_\theta \Rightarrow H_\lambda^2 \approx R_{2\theta}$$

where R_θ is a rotation of angle θ in the plane. Since θ is not a multiple of π , H_λ^2 has no eigenvalues. \square

Now we go back to the structure of the eigenvalues of the double evolute T_P .

5.2 Eigenvalue 0

We already know from Eq. 11 that

$$\langle T_P r, r \rangle = \sum_{i=1}^{2n} \beta_i (r_i - r_{i+1})^2$$

so $T_P \mathbf{r} = 0$ if and only if $\mathbf{r} = \gamma \mathbf{1}$. In other words, $\text{Ker}(T_P) = B_P$, and 0 is always a single eigenvalue of T_P .

5.3 Double eigenvalue 1

Surprisingly, we can state very explicitly a pair of eigenvectors associated to the 1 eigenvalue of T_P in the general case:

Proposition 17 *Let X be any non-zero fixed vector in \mathbb{R}^2 . Then the vector u defined by*

$$u_i = [X, Q_i] \quad (i = 1, 2, \dots, 2n)$$

satisfies $T_P u = u$. Moreover, the P -polygon determined by u does not close, that is, $\mathbf{u} \notin C_P$.

Proof Using our previous notation and remembering 2, we calculate directly

$$\begin{aligned} s_i &= \alpha_i \Delta_i r = \alpha_i [X, \Delta_i Q] = -[X, P_{i+1}] \\ t_i &= -\beta_i \nabla_i s = \beta_i [X, \Delta_i P] = [X, Q_i] \end{aligned}$$

Now the "drift" of the P -polygon after one turn (see Eqs. 6 and 2) is

$$D = \sum_{i=1}^{2n} u_i \Delta_i P = \sum_{i=1}^{2n} \frac{[X, Q_i]}{\beta_i} Q_i.$$

So the component of the drift in the direction orthogonal to X is

$$[X, D] = \sum_{i=1}^{2n} \frac{[X, Q_i]^2}{\beta_i}$$

which is clearly positive, since $\beta_i > 0$ for all i and $X \neq \mathbf{0}$. In other words, the "drift" cannot be $\mathbf{0}$, and the P -polygon does not close. \square

Since there are 2 degrees of freedom in the choice of X , that gives us a 2 dimensional space of eigenvectors. More explicitly, one can define vectors u' and u'' by taking

$$\begin{aligned} u'_i &= [Q_1, Q_i] \\ u''_i &= [Q_2, Q_i] \end{aligned}$$

which are clearly linearly independent: $u' = (0, [Q_1, Q_2], \dots)$ and $u'' = ([Q_2, Q_1], 0, \dots)$. Also, from the symmetry of Q , we see that such solutions satisfy $u_{i+n} = -u_i$, so we have

$$\text{Ker}(T_P - I) \subseteq A_P$$

Actually, consider the following general discrete Sturm-Liouville equation presented in [8]:

$$\nabla_i (a_i \Delta_i u) + (\lambda b_i - c_i) u_i = 0 \quad (16)$$

where $a_i, b_i > 0$ and c_i are given n -periodic sequences and we want to find the eigenvalue λ and the $2n$ -periodic sequence u . A class of such equations can be interpreted as cycloid problems:

Proposition 18 *Suppose the problem described above has $c_i = 0$ and a double eigenvalue 1. Then there is a symmetric, locally convex ball P such that*

$$a_i = \alpha_i = \frac{1}{[Q_i, Q_{i+1}]} \text{ and } b_i = \frac{1}{\beta_i} = [P_i, P_{i+1}]$$

so Eq. 16 becomes the cycloid problem in Eq. 12.

Proof Let $u = x$ and $u = y$ be two linearly independent solutions of 16 associated to the eigenvalue 1. Take the sequence of points $Q_i = (x_i, y_i)$ in the plane. Looking at one coordinate at a time, it is easy to see that

$$\nabla_i (a_i \Delta_i Q) = -b_i Q_i \Rightarrow \nabla_i a \cdot \Delta_i Q - a_{i-1} \cdot \delta_i^2 Q = -b_i Q_i$$

so, remembering that

$$[\delta_i^2 Q, \Delta_i Q] = [\Delta_i Q - \nabla_i Q, \Delta_i Q] = [\Delta_i Q, \nabla_i Q]$$

we can write

$$a_{i-1} [\delta_i^2 Q, \Delta_i Q] = b_i [Q_i, \Delta_i Q] \Rightarrow a_{i-1} [\Delta_i Q, \nabla_i Q] = b_i [Q_i, Q_{i+1}] \quad (17)$$

Note that, if we had $[\Delta_i Q, \nabla_i Q] = 0$ above, that would force $[Q_i, Q_{i+1}] = 0$, which is not possible since x and y are linearly independent. Now, aiming towards Eq. 2, we define

$$P_i = -\frac{\nabla_i Q}{[Q_{i-1}, Q_i]}$$

so we can compute

$$[\nabla_i Q, \Delta_i Q] = [P_i, P_{i+1}] \cdot [Q_{i-1}, Q_i] \cdot [Q_i, Q_{i+1}]$$

That allows us to rewrite Eq. 17 as

$$a_{i-1} [Q_{i-1}, Q_i] = \frac{b_i}{[P_i, P_{i+1}]},$$

so, rescaling Q if necessary, we may assume $a_{i-1} = \alpha_{i-1} = \frac{1}{[Q_{i-1}, Q_i]}$ and $b_i = \frac{1}{\beta_i} = [P_i, P_{i+1}]$, as claimed. Since the sequences a_i and b_i are positive and n -periodic, we can now see that P and Q are locally convex and symmetric. \square

5.4 No triple eigenvalues

Proposition 19 *If λ is any eigenvalue, then*

$$\dim(Ker(T_P - \lambda I)) \leq 2$$

Proof This is a direct consequence of the recurrence – given λ , once the values r_1 and r_2 are chosen, recurrence 14 determines all other coordinates of \mathbf{r} . Or, in other words, H_λ^2 has at most 2 eigenvectors associated to the eigenvalue 1. \square

5.5 No eigenvalues with eigenvectors in both S_P and A_P

Proposition 20 *The restrictions $T_P|_{A_P}$ and $T_P|_{S_P}$ have no common eigenvalues.*

Proof If a eigenvalue λ had eigenvectors in both A_P and S_P , the corresponding half-way transform H_λ would have both 1 and -1 as eigenvalues. Since $\det H_\lambda = 1$, this is impossible. \square

Putting all pieces together, we have the main result of this section:

5.6 The spectral structure of the double evolute

Proposition 21 *The eigenvalues of T_P (in Eq. 10) can be ordered as*

$$\lambda_0 (= 0) < \lambda_1^1 = \lambda_1^2 (= 1) < \lambda_2^1 \leq \lambda_2^2 < \lambda_3^1 \leq \lambda_3^2 < \dots \lambda_{n-1}^1 \leq \lambda_{n-1}^2 < \lambda_n \quad (18)$$

where $\lambda_k^{1,2}$ are eigenvalues of $T_P|_{A_P}$ if k is odd and 0 and $\lambda_k^{1,2}$ are eigenvalues of $T_P|_{S_P}$ if k is even.

Proof Any P -ball can be continuously deformed towards a $2n$ -regular polygon (being kept convex and symmetric in the process). Throughout the process, the eigenvalues of T_P change continuously, and its eigenvectors (always in A_P or S_P) can also be chosen to change continuously. Now, the proposition above guarantees that eigenvalues corresponding to eigenvectors in distinct spaces (one in A_P , another in S_P) cannot "switch places" through the process! Therefore, the ordering of the eigenvalues of T_P (belonging to different spaces) must be the same as it is in the case of regular polygons! \square

Remark 22 The above proposition begs the following question: can one hear the shape of a convex symmetric body²? We mean, given a specific list of cycloid eigenvalues, are we able to determine the shape of the P -ball? Or, a slight variation on this question: if all eigenvalues are double, do we necessarily have a regular polygon?

Remark 23 Note that Eq. 7 can be rewritten in terms of the double evolute transform!

$$\mathbf{r} = (I - T_P) \mathbf{h}$$

So, let \mathbf{r} be the radii associated to a **closed** cycloid, say, $T_P \mathbf{r} = \lambda \mathbf{r}$ with $\lambda \neq 1$. Taking $\mathbf{h} = \frac{1}{1-\lambda} \mathbf{r}$, we define a P -polygon whose radii vector is exactly \mathbf{r} , since

$$(I - T_P) \mathbf{h} = \frac{1}{1-\lambda} \mathbf{r} - \frac{\lambda}{1-\lambda} \mathbf{r} = \mathbf{r}.$$

This shows that the support function of a (correctly placed in the plane) closed cycloid is also an eigenvector of T_P . In other words, any closed P -polygon can be written as the Minkowski sum of $2n - 2$ closed cycloids! Now, periodic support functions cannot represent open polygons like our open cycloids, hence our choice of primarily working with curvature radii.

5.7 Cusps

Definition 24 Given a periodic P -polygon M represented by the radius vector \mathbf{r} , the *orientation* of its side $\Delta_i M$ is the sign of the corresponding radius r_i . A vertex V of M is a *cusp* if its neighbor (non-degenerate) sides have opposite orientations. Such cusp will be named *ordinary* if there is at most one degenerate side at V .

Such cusps have appeared at [5] (where they were called *strong corners*). Geometrically, sides which meet at cusp are on opposite sides of the normal line $\{M_i + tP_i; t \in \mathbb{R}\}$; algebraically, each cusp corresponds to a zero-**crossing** of the sequence $(r_i)_{i \in \mathbb{Z}}$. For example, a snippet $(\dots, -2, 0, 0, 0, 0, 3, \dots)$ corresponds to **one** non-ordinary cusp, while $(\dots, 2, 0, 2, \dots)$ is not a cusp at all.

Now, consider how the number of zero-crossings (per period) of a sequence \mathbf{r} can change if \mathbf{r} is changed continuously. One can create two cusps going from a "+0+" subsequence to "+-+" (or from "-0-" to "-+-", of course); one can destroy two cusps reversing this process. Finally, many zero-crossings can be created at once if a sequence of consecutive 0s is present (for example, from "+000+" to "+-+-+"). Outside of these situations, there is no way to create or destroy a cusp (it is possible to *move it*, of course, going from "++-" to "+0-" to "+--", but in each of these cases we have only one ordinary cusp). Under this light, the following proposition is important:

Proposition 25 *In a cycloid, any zero entries in the radius vector must correspond to cusps; also, all cusps are ordinary.*

Proof This is a direct consequence of the recurrence 14: if $v_k = 0$, then v_{k+1} and v_{k-1} must have opposite signs; if two consecutive sides were 0, all of them would be 0. \square

Proposition 26 *Given a P -ball with $2n$ sides, the number of cusps of its associated cycloids is respectively*

$$0, 2, 2, 4, 4, 6, 6, \dots, 2n - 2, 2n - 2, 2n$$

² Ok, there is no physical *hearing* in this context, but we wanted to cite [4].

Proof Once again, deform P continuously towards a $2n$ -regular polygon. Since ordinary cusps are stable with relation to changes in the radii vectors (as long as no consecutive zeroes occur), the previous proposition guarantees that each cycloid will have a constant number of cusps as the deformation takes place. Now it is just a matter of checking how many cusps each of the eigenvectors in Equation 13 has. \square

6 Other periods

In the Euclidean plane, hypocycloids and epicycloids might take several turns to close. This suggests we could relax the periodicity condition (Eq. 5) on the curvature representation $(r_i)_{i \in \mathbb{Z}}$ of the polygonal line M – requiring, instead, the sequence to be periodic with period $2mn$, say. Can we find discrete closed cycloids with other periods this way?

We claim that a big part of the analysis in such cases is already done! After all, we did not really use that the unit ball P is a **simple** closed convex polygon – we only needed local convexity, as seen in Eqs. 4, so we could establish the positivity of α_i and β_i (defined in Eq. 3). In other words, if one wants to find $2mn$ -periodic cycloids with reference to a $2n$ -ball P , one can instead look for $2mn$ -periodic cycloids with reference to the $2mn$ -ball which is determined by P **traversed m times** (call this polygon mP).

So the reader will have the pleasure of re-reading this article from the beginning switching P with mP , a polygon which goes m times around the origin (two articles for the price of one!). All calculations in Section 2 are unchanged, except that indices go $i = 1, 2, \dots, 2mn$. The new curvature radius space of Section 3 (that would be L_{mP}) has dimension $2mn$, and the corresponding spaces S_{mP} and A_{mP} are still orthogonal complements of each other. All calculations done in Section 4 still hold, but the matrices are $2mn \times 2mn$. Finally, all arguments in Section 5 still hold, with a few exceptions – first, our base case must change³:

Example 27 (Regular Polygon traversed m times) If P is a regular polygon with $2n$ sides, traversed m times, then so is Q . Write $\alpha = \frac{\pi}{2n}$ and $\gamma = \frac{\pi}{2mn}$ and assume by rescaling that

$$|P_i| = |Q_i| = \frac{1}{\sqrt{\cos \alpha}} \Rightarrow \alpha_i = \beta_i = \frac{1}{2 \sin \alpha}.$$

The double evolute transform is the same as before, except for the matrix size which now must be $2mn \times 2mn$:

$$T_{mP} = \frac{1}{4 \sin^2 \alpha} \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ -1 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}$$

The eigenvectors are

$$\mathbf{d}_k = (1, \cos 2k\gamma, \cos 4k\gamma, \dots, \cos 2(2mn - 1)k\gamma); \quad k = 0, 1, 2, \dots, 2mn - 1$$

³ In fact, all the theory could be done if P were any locally convex polygon that goes around the origin m times (not necessarily repeating itself at each turn), but then the geometric interpretation of P as a unit ball is somewhat diminished. **Three** articles for the price of one!

and the eigenvalues are

$$\sigma_k = \frac{\sin^2 k\gamma}{\sin^2 \alpha}$$

for $k = 0, 1, 2, \dots, 2mn - 1$. Again, $\sigma_k = \sigma_{2mn-k}$, so all of them are double eigenvalues, except for $\sigma_0 = 0$ and $\sigma_{mn} = \csc^2 \alpha$. Renaming $(\sigma_k, \sigma_{2mn-k})$ as (σ_k^1, σ_k^2) for $k = 1, 2, \dots, mn - 1$, the eigenvalues can be ordered this way:

$$\sigma_0 (= 0) < \sigma_1^1 = \sigma_1^2 < \sigma_2^1 = \sigma_2^2 < \dots < \sigma_m^1 = \sigma_m^2 (= 1) < \dots < \sigma_{mn-1}^1 = \sigma_{mn-1}^2 < \sigma_{mn} (= \csc^2 \alpha)$$

Each eigenvector has only regular cusps. In fact, taking $k = 0, 1, 2, \dots, mn$, we can see that the number of cusps in \mathbf{d}_k is exactly $2k$. So the number of cusps in each cycloid can be ordered (correspondingly to the eigenvalues) in the list

$$0, 2, 2, 4, 4, \dots, 2mn - 2, 2mn - 2, 2mn.$$

Finally, if $k = jm$ ($j = 1, 2, \dots, n$) then $k\gamma = j\alpha$, and

$$\mathbf{d}_{jm} = \mathbf{c}_j \text{ and } \sigma_{jm}^{1,2} = \lambda_j^{1,2}$$

where λ and \mathbf{c} are the eigenvalues/vectors in the case the polygon was traversed just once (see Example in page 12).

So the following (partial!) result is easily obtained as before:

Proposition 28 *When the unit ball is traversed m times, the eigenvalues of the double evolute transform can be ordered as*

$$\sigma_0 (= 0) < \sigma_1^1 \leq \sigma_1^2 < \sigma_2^1 \leq \sigma_2^2 < \dots < \sigma_m^1 = \sigma_m^2 = 1 < \dots < \sigma_{mn-1}^1 \leq \sigma_{mn-1}^2 < \sigma_{mn}$$

where $\sigma_k^{1,2}$ are eigenvalues of $T_{mP}|_{A_{mP}}$ if k is odd and 0 and $\sigma_k^{1,2}$ are eigenvalues of $T_{mP}|_{S_{mP}}$ if k is even. The number of cusps of the associated cycloids are respectively

$$0, 2, 2, 4, 4, \dots, 2mn - 2, 2mn - 2, 2mn.$$

Proof As before, start with a $2n$ -regular polygon, traversed m times around the origin, and deform it continuously towards mP . Since the eigenvalues $\sigma_k^{1,2}$ cannot switch places with $\sigma_{k+1}^{1,2}$ (that would imply an eigenvalue common to A_{mP} and S_{mP}), the ordering above must be kept throughout. Similarly, since all cusps are kept ordinary throughout the deformation, their number must be constant in each eigenvector. \square

6.1 From one turn to many turns

Our final goal this section is to relate the eigenvalues of T_{mP} with the eigenvalues of T_P . To do that, we fully turn our attention to the half-turn transform, for now we have:

$$\begin{bmatrix} r_{2mn+2} \\ r_{2mn+1} \end{bmatrix} = H_\lambda^{2m} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

So now we say:

- "Finding an eigenvector $\mathbf{r} = (r_1, r_2, \dots, r_{2mn-1}) \in L_{mP}$ of T_{mP} (for the eigenvalue λ)" is equivalent to
- "finding an eigenvector $\boldsymbol{\rho} = (r_1, r_2) \in \mathbb{R}^2$ of H_λ^{2m} (for the eigenvalue 1)";

Adapting proposition 16, we have:

Proposition 29 *Let $\lambda \in \mathbb{R}$. We can once again classify λ as an eigenvalue of T_{mP} according to the **geometric** multiplicity of the eigenvalues of H_λ :*

Case 1. If $H_\lambda = I$, then λ is a double eigenvalue of T_{mP} , and both its eigenvectors are in S_{mP} .

Case 2. If $H_\lambda = -I$, then λ is a double eigenvalue of T_{mP} , and both its eigenvectors are in A_{mP} .

Case 3. If H_λ has a single eigenvalue 1, then λ is a single eigenvalue of T_{mP} , with its eigenvectors in S_{mP} .

Case 4. If H_λ has a single eigenvalue -1 , then λ is a single eigenvalue of T_{mP} , with its eigenvectors in A_{mP} .

Case 5. If H_λ has two distinct real eigenvalues μ and μ^{-1} , then λ is not an eigenvalue of T_{mP} .

*Case 6a. If H_λ has two complex eigenvalues $e^{\pm i\theta}$ and θ is an even(odd) multiple of $\frac{\pi}{m}$, then λ is a **double** eigenvalue of T_{mP} , with its eigenvectors in $S_{mP}(A_{mP})$.*

Case 6b. If H_λ has two complex eigenvalues $e^{\pm i\theta}$ and θ is not a multiple of $\frac{\pi}{m}$, then λ is not an eigenvalue of T_{mP} .

Proof Cases 1 and 2 are as before. Cases 3 and 4 are also very much the same, since

$$H_\lambda \approx \begin{bmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{bmatrix} \Rightarrow H_\lambda^{2m} \approx \begin{bmatrix} 1 & \pm 2m \\ 0 & 1 \end{bmatrix}$$

which clearly has only one eigenvalue 1. In case 5 the eigenvalues of H_λ^{2m} are μ^{2m} and μ^{-2m} , and neither is 1. Now, in case 6:

$$H_\lambda \approx R_\theta \Rightarrow H_\lambda^{2m} \approx R_{2m\theta}$$

and that is why we have to separate it further: if $2m\theta$ is a multiple of π , then $H_\lambda^{2m} = \pm I$ and we are back to cases 1 or 2; otherwise, H_λ^{2m} has no eigenvalues. \square

Do note that the half-turn transform H_λ is exactly the same here as it was in the "one turn" case! So cases 1-5 happen just as often here as they did before, and with the same values for λ – they account for $2m$ of the $2mn$ eigenvalues we found! So all the new eigenvalues must come from case 6a... Can we figure out their ordering with relation to the "old" eigenvalues? Indeed we can – we just need another continuity argument.

Proposition 30 *Let the specter of T_P be as denoted in 18. The eigenvalue(s) μ (and μ^{-1}) of H_λ depend on λ in the following way:*

- a) *If $\lambda_k^2 < \lambda < \lambda_{k+1}^1$, then H_λ has two complex eigenvalues (as in case 6). In fact, as λ grows from λ_k^2 to λ_{k+1}^1 , the eigenvalues μ and μ^{-1} go through all values in the complex unit circle exactly once.*
- b) *If $\lambda_k^1 < \lambda < \lambda_k^2$ or $\lambda < 0$, then H_λ has two distinct real eigenvalues (as in case 5).*

Proof Just consider the positioning of μ in the complex plane in each of the 6 cases we considered, as displayed in Figure 6:

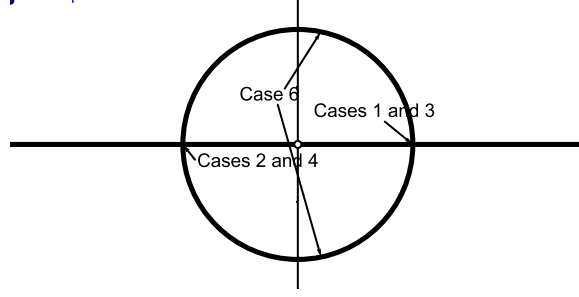


Fig. 6 Where is μ in the complex plane?

Since μ varies continuously with λ (and $\mu \neq 0$), in order to go from cases 1, 3 to 2, 4 (and vice-versa) we must go through all complex numbers in the circle $|\mu| = 1$ (they do come in conjugate pairs, of course). So each interval of the kind $I_k = (\lambda_k^2, \lambda_{k+1}^1)$ will contain at least $m - 1$ values of λ which correspond to the eigenvalues $e^{\pm i\theta}$ for $\theta = \frac{k\pi}{m}$ ($k = 1, 2, \dots, m - 1$) of H_λ , as in case 6a. Now, how do we know that each of these complex eigenvalues is visited only once as λ varies in I_k ? Every time λ falls into case 6a, such λ is an eigenvalue of T_{mP} . Since there are n such intervals I_k , and each interval already contains $m - 1$ **double** eigenvalues, this already accounts for $2(m - 1)n$ eigenvalues. If we now add the $\lambda_k^{1,2}$ themselves (there are $2n$ of them), which are also eigenvalues of T_{mP} , we are already crowded with all $2mn$ eigenvalues T_{mP} could possibly have! So no other values of λ can generate eigenvalues of H_λ of the form $e^{\pm i\theta}$ where θ is **any** rational multiple of π . That proves not only that each complex eigenvalue is visited only once in each interval (a), but also shows that in (b) no new complex eigenvalues μ can appear – so while $\lambda \in (\lambda_k^1, \lambda_k^2)$ or $\lambda < 0$, we must keep μ real and different from 1. \square

We can now gather all information we have in one final proposition:

Proposition 31 *The $2mn$ eigenvalues of T_{mP} can be ordered the following way*

$$\begin{array}{llll}
 & & & \lambda_0 (= 0) < \\
 < \lambda_{1/m}^1 = \lambda_{1/m}^2 & < \lambda_{2/m}^1 = \lambda_{2/m}^2 & < \dots < \lambda_{1-1/m}^1 = \lambda_{1-1/m}^2 < \lambda_1^1 = \lambda_1^2 (= 1) < \\
 < \lambda_{1+1/m}^1 = \lambda_{1+1/m}^2 & < \lambda_{1+2/m}^1 = \lambda_{1+2/m}^2 & < \dots < \lambda_{2-1/m}^1 = \lambda_{2-1/m}^2 < \lambda_2^1 \leq \lambda_2^2 < \\
 < \lambda_{2+1/m}^1 = \lambda_{2+1/m}^2 & < \lambda_{2+2/m}^1 = \lambda_{2+2/m}^2 & < \dots < \lambda_{3-1/m}^1 = \lambda_{3-1/m}^2 < \lambda_3^1 \leq \lambda_3^2 < \\
 < \dots & \dots & \dots & \dots < \\
 < \lambda_{n-1+1/m}^1 = \lambda_{n-1+1/m}^2 < \lambda_{n-1+2/m}^1 = \lambda_{n-1+2/m}^2 < \dots < \lambda_{n-1/m}^1 = \lambda_{n-1/m}^2 < \lambda_n
 \end{array}$$

Proof All the work is already done – just define $\lambda_p^{1,2}$ as the value of λ which makes H_λ have the eigenvalues $e^{\pm i\pi p}$ when the integer part of p is even (that is, when you are moving from cases 1, 3 to 2, 4); or $e^{\pm i\pi(1-p)}$ otherwise. \square

6.2 No periods!

What if we relax the periodicity condition even further: let us **not** require the list $\{r_n\}_{n \in \mathbb{Z}}$ to be periodic. What then?

First of all, clearly the eigenvalues can now be any real number. After all, just pick any $\lambda \in \mathbb{R}$, any two values $r_1, r_2 \in \mathbb{R}$ and apply recurrence 14 both backwards and forwards to create the complete list $\{r_n\}$. Moreover, if the number λ you picked is any of the eigenvalues of T_{mP} for *some* m , the list will be periodic of period $2mn$, as seen above (and, unless $\lambda = 1$, the cycloid will close). Otherwise, we must have non-periodic cycloids! As such, one interesting phenomenon (which does not exist in the Euclidean case) can occur – a *spiraling cycloid*.

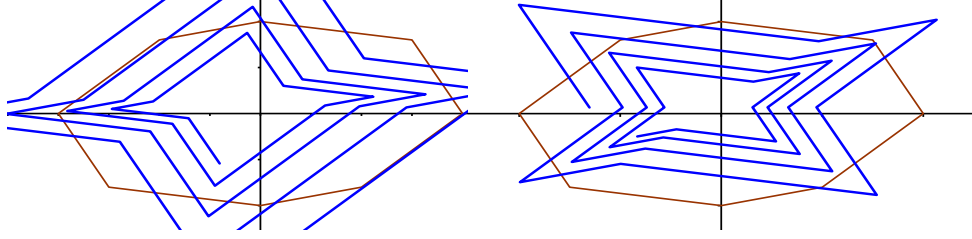


Fig. 7 Two very different spiraling cycloids associated to the same $\lambda \approx 3.3$

Proposition 32 Suppose $\lambda_n^1 < \lambda < \lambda_n^2$ or $\lambda < 0$. Then the cycloids associated to λ are unlimited.

Proof Just remember that in this case, the eigenvalues associated to H_λ can be written as μ and μ^{-1} where $|\mu| > 1$. Writing $\boldsymbol{\rho} = (r_1, r_2) = c_1 \rho_1 + c_2 \rho_2$ where ρ_1 and ρ_2 are the respective eigenvectors of H_λ we have $H_\lambda^m \boldsymbol{\rho} = c_1 \mu^m \rho_1 + c_2 \mu^{-m} \rho_2$. If $c_1 \neq 0$, we have $|r_{mn+1}| \rightarrow \infty$ as $m \rightarrow \infty$; if $c_1 = 0$, then $|r_{mn+1}| \rightarrow \infty$ as $m \rightarrow -\infty$. Either way, the cycloid is unlimited. \square

7 A Four Vertex Theorem

In differential geometry, a “*vertex*” of a curve is a point where the curvature reaches a local extremum. We want to show that any closed P -polygon has at least 4 vertices... of this other kind, which needs to be defined.

Definition 33 An *edgex* of a P -polygon is a collection of adjacent sides whose (equal) curvatures correspond to a strict extremum of the sequence of radii in \mathbf{r} . In other words, an edgex is a zero-crossing of $\Delta \mathbf{r}$.

To clarify, if the radii vector is $(\dots, 1, 2, 3, 3, 3, 2, 2, \dots)$, we count that sequence of threes as **one** edgex, but if the sequence were $(\dots, 1, 2, 3, 3, 3, 4, 5, \dots)$ we see no edgex at all in this part of the polygon. So we are ready to state our “four edgex theorem”, which adapts the reasoning in [6]:

Proposition 34 Any closed convex polygon M (which is not homothetic to the P -ball) has at least four edgices. If M has constant P -width, it must have at least six edgices.

Proof Take the radii vector $\mathbf{r} \in C_P$ associated to M , and decompose it as

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{r}_2^1 + \mathbf{r}_2^2 + \mathbf{r}_3^1 + \mathbf{r}_3^2 + \dots + \mathbf{r}_{n-1}^1 + \mathbf{r}_{n-1}^2 + \mathbf{r}_n$$

where each \mathbf{r}_i^j is a cycloid associated to the eigenvalue λ_i^j (note that $i \neq 1$; suppose for now that $\mathbf{r}_2^1 \neq \mathbf{0}$). Since \mathbf{r}_0 is a multiple of $\mathbf{1}$, it does not alter its number of local extrema, so we may discard it completely. We can define a *double involute* of M as the P -polygon with radii vector given by

$$I_P \mathbf{r} = \frac{\mathbf{r}_2^1}{\lambda_2^1} + \frac{\mathbf{r}_2^2}{\lambda_2^2} + \dots + \frac{\mathbf{r}_n}{\lambda_n}$$

Note that $T_P I_P \mathbf{r} = \mathbf{r}$ (I_P is the pseudo-inverse of T_P). Now the key to the proof is to realize that every application of either evolute transform E_P or E_Q cannot decrease the number of edgices! This should be clear from Eq. 9 – since $s_i = \alpha_i \Delta_i r$, each edge of \mathbf{r} must correspond to a zero crossing of \mathbf{s} , and between two consecutive zero crossings of \mathbf{s} we must have an edgex of \mathbf{s} ! In other words, since \mathbf{r} is the double evolute of $I_P \mathbf{r}$, we conclude that \mathbf{r} must have at least as many edgices as $I_P \mathbf{r}$. Now iterate I_P ! So \mathbf{r} has at least as many edgices as

$$I_P^k \mathbf{r} = \frac{\mathbf{r}_2^1}{(\lambda_2^1)^k} + \frac{\mathbf{r}_2^2}{(\lambda_2^2)^k} + \dots + \frac{\mathbf{r}_n}{(\lambda_n)^k} = \frac{1}{(\lambda_2^1)^k} \left(\mathbf{r}_2^1 + \left(\frac{\lambda_2^1}{\lambda_2^2} \right)^k \mathbf{r}_2^2 + \dots + \left(\frac{\lambda_2^1}{\lambda_n} \right)^k \mathbf{r}_n \right)$$

We might as well ignore the homothety of a factor of $(\lambda_2^1)^k$, and note that eventually this involute will be arbitrarily close to \mathbf{r}_2^1 – which is a cycloid with 4 cusps, and therefore 4 edgices (if $\lambda_2^1 = \lambda_2^2$, just group together \mathbf{r}_2^1 and \mathbf{r}_2^2 to form a single cycloid with 4 cusps and repeat the argument). If it so happens that $\mathbf{r}_2^1 = \mathbf{r}_2^2 = \mathbf{0}$, just repeat the argument using the first non-zero cycloid instead of \mathbf{r}_2^1 , and the number of cusps will be even bigger. For example, if the initial curve has constant P -width, then it must live in $W_P = B_P \oplus D_P$; since we are ignoring the component \mathbf{r}_0 , we have a vector in $D_P \subseteq A_P$, so the \mathbf{r}_2 components **must** be zero and the decomposition starts with \mathbf{r}_j^i where $j \geq 3$ – a cycloid with 6 cusps or more, and therefore 6 edgices or more. \square

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